THE BROKEN SPAGHETTI NOODLE

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A popular question in recreational mathematics is the following: If we drop a spaghetti noodle and it breaks at two random places, what is the probability that we can form a triangle with the three resulting segments? See for example [**2**, chap.1, sec.6], [**3**, p.6], [**4**, p.31], or [**7**, p.30-36]. This is an elementary problem in geometric probability. Clearly the length of the noodle (or equivalently our choice of unit length) does not matter, so the problem amounts to choosing two numbers at random from the interval $(0, 1)$, say a and b with $a < b$, and looking at the resulting intervals $(0, a)$, (a, b), and $(b, 1)$. We will be able to form a triangle when the positive numbers a, $b-a$, and $1-b$ satisfy the triangle inequality (i.e., when no interval is longer than the combined lengths of the other two). Equivalently, this will be the case when all three intervals have length less than $1/2$. Therefore, a triangle can be formed precisely when the following three inequalities hold: $a < 1/2$, $b - a < 1/2$, and $b > 1/2$.

Figure 1 shows all possible outcomes $0 < a < b < 1$, and the darker shaded region consists of all "favorable" outcomes, when a triangle can be formed. Comparing areas, we see that the probability of succeding in getting a triangle is $1/4$.

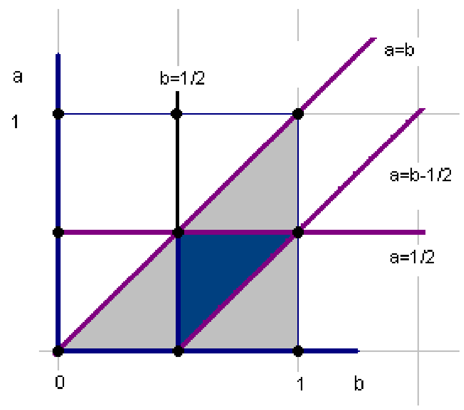

Figure 1.

In this note we solve the following generalization of the problem: If the noodle breaks at $n-1$ random places, what is the probability $P(n)$ that we can form an n-gon with the n resulting segments? Again, this can be modeled by choosing $n-1$ numbers at random from $(0, 1)$, say $a_0 = 0 < a_1 <$

$a_2 < \cdots < a_{n-1} < 1 = a_n$, and looking at the resulting intervals (a_{i-1}, a_i) for $i = 1, 2, \ldots, n$. Setting $x_i = a_i - a_{i-1}$, we have an affine isomorphism between the $(n-1)$-tuples $(a_1, \ldots, a_{n-1})$ satisfying the foregoing string of inequalities and the $(n-1)$-dimensional set Δ_n in $\mathbb{R}^n$ given by

$$\Delta_n = \{(x_1, \ldots, x_n) : \text{each } x_i > 0,\ \sum_{i=1}^{n} x_i = 1\}.$$

This is the set of "all possible outcomes." The set of all "favorable" outcomes is given by the following proposition. Consider the subset Υ_n of Δ_n defined by

$$\Upsilon_n = \{(x_1, \ldots, x_n) \in \Delta_n : x_i < 1/2 \text{ for } i = 1, 2, \ldots, n\}.$$

Proposition 1. *There exists an n-gon of perimeter 1 and side-lengths $x_1, \ldots, x_n$ if and only if $(x_1, \ldots, x_n)$ lies in Υ_n.*

Proof. Suppose that $(x_1, \ldots, x_n) \notin \Upsilon_n$, say $x_k \geq 1/2$ for some k. Then $\sum_{i \neq k} x_i \leq 1/2$ and it is impossible to form an n-gon. Conversely, if the length of the longest side (hence of all sides) is less than $1/2$, then the sum of the lengths of the other sides is larger than $1/2$ and a little tweaking yields an n-gon. $\square$

We record as a simple consequence of Proposition 1:

Corollary 2 (Generalized Triangle Inequality). *Let $y_1, \ldots, y_n$ be positive numbers. There exists an n-gon with side-lengths $y_1, \ldots, y_n$ if and only if*

$$y_i \leq \sum_{j \neq i} y_j \quad (i = 1, \ldots, n).$$

We remark that we should expect $P(n) \to 1$ as $n \to \infty$ because, given any $\epsilon > 0$, the probability that the longest side will have length greater than or equal to ϵ tends to zero as n tends to infinity.

It is clear that $P(n) = \mu(\Upsilon_n)/\mu(\Delta_n)$, where μ is any $(n-1)$-dimensional Euclidean measure on the subsets of Δ_n. One could use such a measure to find $P(n)$: after computing $\mu(\Delta_n)$, one partitions Δ_n into Υ_n and other pieces, just as we do in the proof of Theorem 3. A similar computation then yields $\mu(\Upsilon_n)$. However, because of the geometry of the pieces of this partition, there is no need at all for computations using any explicit measure. Here is our main result:

Theorem 3. *The probability $P(n)$ is given by*

$$P(n) = 1 - \frac{n}{2^{n-1}}.$$

Note that this formula is consistent with our previous result for $P(3)$ and also with the remark that $\lim_{n \to \infty} P(n) = 1$.

Proof. Let $D_i = \{(x_1, \ldots, x_n) \in \Delta_n : x_i \geq 1/2\}$. We can decompose Δ_n as the disjoint union

$$\Delta_n = \Upsilon_n \cup \bigcup_{i=1}^{n} D_i.$$

Thus Υ_n is obtained by slicing off each "corner" D_i of the simplex Δ_n at the midpoint of each edge. But each D_i is actually similar to Δ_n by a scaling factor of $1/2$, so it has measure $(1/2)^{n-1}$ times the measure of Δ_n. Therefore,

$$\mu(\Upsilon_n) = \mu(\Delta_n) - n(1/2)^{n-1}\mu(\Delta_n) = [1 - n(1/2)^{n-1}]\mu(\Delta_n),$$

and the theorem follows. □

Readers interested in geometric probability might want to see the books [**4**] and [**6**]. Those interested in the random division of an interval can consult the articles [**1**] and [**5**]. Finally, those interested in similar problems can find some in [**2**, chap.1, sec.6], [**3**], and [**7**].

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